Non-reducible Conceptual Frameworks in Mathematics

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Abstract

In this paper we propose that there are two ways in which mathematical concepts can be acquired. The first is the transfer of conceptual frameworks in which the new concept can be completely reduced to those used to define it and their interactions. The second is akin to emergence in which the new concept is not reducible to the prior ones, but arises (or emerges) from the combinations and interactions of different modes of thinking or acting, i.e., different conceptual frameworks. We show that these two ways of understanding and acquiring mathematical concepts have parallels in the context of the philosophical discussion on epistemological reducibility and emergence. As a case study we look at the construction of real numbers as Dedekind cuts based on Zermelo-Fraenkel Set Theory. Finally, we draw on research in cognitive science to illustrate our points and show how they bind to previous research.

> It's more or less standard orthodoxy these days that set theory – ZFC, extended by large cardinals – provides a foundation for classical mathematics. Oddly enough, it's less clear what 'providing a foundation' comes to.
>
> (Maddy, 2017)

1 Introduction

How are mathematical concepts acquired, grasped and communicated? How is the conceptualisation and understanding of mathematical notions such as "real number line", "prime number" and "a basis of a separable Hilbert space" achieved? One answer is that mathematical notions are grounded in their definitions. To avoid infinite regress, it is often assumed that some basic notions[1] do not require other than an intuitive, non-formal definition. In this article we argue that grounding in a formal definition is indeed an important way in which the conceptualisation of mathematical concepts can be achieved. However, it is not sufficient to account for the richness of most mathematical concepts. The main purpose of this paper is to propose another way in which mathematical concepts are grasped which has little to do with their formal definition and which is arguably a necessary part of mathematical practice.

To analyse the question, we introduce the notion of *transfer of conceptual framework* (Section 3). A transfer of conceptual framework occurs when a mathematical concept can

[1] Such as sets, classes, membership relation, natural numbers etc.

be grasped by understanding its definition: the concepts in it, the way they are combined, and nothing else. In this case, the concept *reduces* to the understanding of the prior concepts. Given that in mathematical practice new concepts are always defined using previously understood concepts, it may seem that all mathematical concepts can be are reducible to the the prior notions, all the way down to the basic ones[2] We argue that it is not the case. We argue that analogously to epistemological reducibility and weak emergence (Bedau, 1997; Clark, 1998), mathematical concepts are not always reducible (through the transfer of a conceptual framework) to the more fundamental notions that are used in their definitions, constructions and descriptions. In fact, in a wealth of central mathematical definitions such a transfer is not sufficient to explain the richness of new concepts. Thus, we end up with a division: on the one hand, there are mathematical concepts that are reducible to prior notions, and on the other hand, there are concepts which supervene and transcend the prior notions making their acquisition, grasping and communication a more complex endeavour.

This idea can be elucidated through a thought experiment. Let us imagine that we had to communicate our mathematical ideas to an intelligent alien species. Suppose that we have already established common ground by agreeing about basic mathematical notions such as sets and first-order logic. Does it follow that we can convey all the mathematical concepts and ideas that can be expressed in terms of these basic notions to the aliens? While this might seem to be possible from the point of view of a "set theoretical reductionist", we argue that it is not necessarily the case. The richness of mathematical ideas can require grasping content that is not present on the level of the basic notions. For instance, if we can reliably communicate our idea of natural numbers, then we can communicate the concept of prime numbers, because the transfer of conceptual framework occurs. But we will argue that even if we can communicate the basic notions of set theory, we might still not be able to communicate the concept of the real number line, because of a different conceptual frameworks. We will argue that even though the reals can be constructed in set theory as Dedekind cuts, this would not be sufficient to conveying to the aliens what we mean by the real numbers.

Drawing on the analysis of Maddy (2017), we argue that while the Zermelo-Fraenkel set theory (ZFC) has multiple facets of a good foundation, it lacks the capacity to explain, talk about, or capture any other concepts than those present in the conceptual framework of sets. In this way, we show that the concept of the real number line which emerges in the Dedekind cut construction is not epistemologically reducible to the axioms and fundamental notions of set theory. Something else is needed to understand and communicate it.

Since our contention is that mathematical concepts are not always epistemologically reducible to the lower-level concepts used in their definition, this prompts the question what the mechanism of emergence of mathematical concepts *is*. In Section 6 we review various approaches in cognitive science and philosophy that attempt to explain the emergence everyday concepts as grounded in the sensorimotor interactions as well as interaction between different sensorimotor domains (Barsalou, 2008; Noë, 2004). We then suggest that

[2]This way of thinking was probably also behind the "New Math" movement in the 1960's – the idea that mathematical concepts can be most fruitfully communicated, in this case to children, by building them up from primitive concepts.

similar principles are feasibly behind more abstract concept formation – such as that in mathematics. We argue that the embodiment-based theories of meaning and concept acquisition operate on analogous principles as the theories of mathematical cognition as proposed by Lakoff and Núñez (2000); Pantsar (2015). In this way, we attempt to bridge together the emergence of meaning in mathematics with general meaning acquisition. Finally, we justify the position by drawing on examples in mathematics – the notions of symmetry and computability – as well as developmental cognitive science.

2 Reducibility and Emergence

Among philosophers, there is a great deal of disagreement on what is meant by emergence and reducibility. The usual starting point is that emergent properties arise from lower-level properties, but are not reducible to them. In this way, reducibility can be seen as contradicting emergentism. Historically, consciousness has been seen as one subject that divides philosophers in this regard. Materialists believe that consciousness is reducible to the physical interactions in the brain, whereas more dualistically inclined philosophers often accept that while consciousness may indeed emerge from the brain, it is not reducible to it as a physical system.

In some cases, the distinction between reducibility and emergence is quite clear. Under a very strict notion of reducibility, we can say that a concept or a phenomenon is reducible to a set of other concepts or phenomena if it is *identical* to them taken separately. For example, the concept of "five pens" is reducible to "one pen", "one pen" and "three pens". It is clear that "five pens" is not emergent in any interesting sense. However, as we shall see, there are cases in which the question of reducibility and emergence is not at all clear.

Perhaps the best candidate for "the standard view" on ontological emergence is the so-called "supervenient emergentism". Under this understanding of emergence, if higher-level emergent properties are altered, then by (nomological) necessity so are some of the lower-level properties (Van Cleve, 1990; McLaughlin, 1997). Thus supervenient emergentism is committed to downward causation: higher-level causes have lower-level effects. Kim (1999) has presented the famous causal-exclusion argument against such supervenient emergentism of mental states. According to him the supervenient conception of emergence is inconsistent, because it implies that physical effects can be caused twice by distinct causes – physical and mental. Although this argument is not always accepted (see e.g., Kallestrup, 2006), it has led many philosophers to look for less problematic views on emergence.

The best-known of these has been presented by Bedau (1997) and Clark (1998). According to them, to contrast the strong supervenient, *ontological* emergentism, we should also look for weaker, *epistemological* versions of emergence. They point out that the interactions in the lower-level states can be so complicated that reducing the higher-level phenomena to them may be in practice impossible, thus making an exhaustive explanation unfeasible. Well-known examples of this are chaotic systems, whose higher-level dynamics are very sensitive on minuscule lower-level details. But to contrast this kind of weak emergentism with the strong ontological emergentism, in Bedau's definition of weak emergence it is *in principle* possible to derive the higher-level dynamics by simulating the lower-level

Example	Epistemologically emergent	Ontologically emergent
Five pens	No	No
2BP	No	No
3BP	Yes	No
Mpemba	Yes	No
Consciousness	Yes	According to some

Table 1: Examples of phenomena and their status in the epistemological and ontological emergence frameworks.

phenomena. Thus weak emergentism takes its place between reducibility and ontological emergentism as an epistemologically motivated alternative.

To give an example, as of writing this, the "Mpemba effect" in which warm water can freeze faster than cold water is still in need of an explanation. Is freezing an emergent property of water molecules? In the strong ontological sense, this seems implausible. In the epistemological sense, however, emergence is a much more appealing position. At least currently, it is not known how to reduce freezing to known (quantum-) physical properties of water molecules. Thus, it seems that one can coherently believe both in reduction and emergence of the same phenomenon at the same time.

Sometimes epistemological emergence may occur due to computational complexity. To see this, let us consider the "two-body" (2BP) and the "three-body" (3BP) problems in physics. If there is just one body, its dynamics are a straight-forward consequence of Newton's first law of motion, as the body progresses linearly in time. If we do not assume any gravitational interaction between the bodies, the same will be the case with two or three objects. As soon as gravity is included, however, the problem becomes quite different. Whereas the two-body problem can be solved and we can determine the motion of the bodies given initial conditions, with three bodies the problem of predicting their behaviour becomes intractable for most initial conditions (Herman, 1998). This behaviour, however, is still ontologically reducible to the properties of the bodies and their gravitational interactions. So in the ontological sense, there can be no difference between the 2BP and the 3BP. But epistemologically there is an important distinction.

Table 1 shows an overview of epistemological emergence and reducibility through the above examples. It may seem strange that the case of five pens is similar to the two-body problem. Of course this is not to suggest that the problems are equally difficult. Determining whether there are five pens in a collection is a trivial problem whereas 2BP is not, but for the perspective of our approach in this paper, the two problems are essentially the same. The important aspect here is that solving a case of 2BP does not require bringing in higher-order methods: it can be solved by the same principles we use to explain gravity and the movement of objects. In this way, while 2BP is in practice more difficult than the one-body problem, it is not essentially different from it. With the 3BP, it is not only the practical difficulty of the problem that changes. Rather, we need a change in the whole mode of explanation (bring in the language of approximations, chaos theory, topology and so on).

In the next sections we want to apply a similar epistemological distinction between

Example	Epistemologically emergent	Ontologically emergent
Five pens	No	No
2BP	No	No
3BP	Yes	No
Mpemba	Yes	No
Prime number	No	No
$\mathbb{R}$	Yes	No

Table 2: Although it may have seemed like mathematical definitions are always analogous to the definition of five pens, the difference between the cases of prime numbers and the real number line shows us that they differ in terms of epistemological emergence (and thus reducibility) too.

reduction and emergence to mathematical definitions and concepts and argue that sometimes they are reducible to the concepts used to define them and their interactions – analogously to the case of five pens or 2BP – and sometimes not – analogously to Mpemba effect and the 3BP.

3 Transfer and Emergence of Conceptual Frameworks

Let us start by considering the definition of a prime number: "a prime number is a natural number greater than 1 which is divisible only by 1 and by itself". This definition uses a variety of existing concepts and relations to define a new one. These ground notions are referred to by the terms "natural number", "greater than", "divisible" and "1". If all these prior concepts are understood, then they can be integrated in the described way and a new mathematical concept, prime number, naturally arises. In this way, the semantics of the expression "prime number" is reducible to a certain combination of these more basic concepts. Importantly, the *conceptual framework* of the concept "prime number" is the same conceptual framework as that of the basic concepts. By "conceptual framework", we refer to the network of concepts in which a particular concept is situated. In the case of "prime number", the framework is that of arithmetical concepts, i.e., the same framework as that of the concepts "natural number", "greater than", "divisible" and "1".

This is analogous to the case of the two-body problem from the previous section. In the same way as the dynamics of two bodies can be fully understood and predicted from initial conditions, the concept of prime numbers can be fully understood by someone who understands "natural number", "greater than", "divisible" and "1", as well as how they are put together (analogous to the interaction of the bodies through gravity). We say that the conceptual framework *transfers* from the basic, ground, concepts to the new one.

Note, however, that understanding the concept "prime numbers" as defined above is not quite as simplistic as the case of "five pens" in the previous section. Whereas understanding "five pens" is identical to understanding the meaning of "one pen", "one pen" and "three pens", understanding "prime number" is not identical to understanding the meaning of "natural number", "greater than", "divisible" and "1" taken separately. The interaction is important in the same way as it is in the 2BP. Let us clarify what we mean by the analogy here. The bodies in 2BP behave like the basic notions such as "natural number" and "1". The interaction between the bodies (gravity) mirrors the interaction between the basic notions such as "greater than" and "divisible by". The behaviour of complex trajectories of the bodies is thus similar to prime numbers: they are both conceptually more complex than the basic concepts used in their characterization. But in both cases the conceptual framework used in explaining the basic concepts and the new complex concept is the same.

Another example of such transfer of the conceptual framework in mathematics is the definition of a power set. Supposing that we understand the concepts of set and subset (and some predicate logic), we can put these concepts together as "the set of all subsets of a given set". Thereby the concept of power set is reduced to the concepts of "set" and "subset", employing the same conceptual framework of set theory.

If all mathematical concepts and their definitions allowed for the transfer of conceptual framework, all concepts would be reducible to a handful of basic notions. However, just like with the difference between the two-body and three-body problems in physics, we argue that many central mathematical notions cannot be explained by only appealing to transfer of conceptual frameworks. When the conceptual framework does not transfer, the defined concept is epistemologically emergent. When introducing the third body to a physical setting, a qualitative change happens and we can no longer use the same methods to explain the dynamics of the system. Instead of an analytic solution, we must look for numerical methods and approximate solutions. In other words, the conceptual framework of the basic concepts does *not* transfer to the new level: although there is no physical difference between the bodies and gravity in the 2BP and 3BP, the epistemologically emergent characteristics of the latter necessitate a new conceptual framework.

To show how an analogous introduction of a novel conceptual framework angle can occur with mathematical definitions, let us now consider the construction of the real number line ($\mathbb{R}$) through Dedekind cuts of rational numbers. The construction is briefly reviewed in Box 1; a more thorough exposition can be found in standard set theory textbooks (e.g. Enderton, 1977).

A Dedekind cut is a subset of the rational numbers which satisfies a certain simple formula. The system of Dedekind cuts can be equipped with an ordering and operations so that it becomes isomorphic to the ordered field of real numbers. The question is, does the case of $\mathbb{R}$ work like the case of prime numbers and we can use the lower-level concepts (sets) to explain the higher-level one (the real number line)? That is, does the construction of Dedekind cuts imply that $\mathbb{R}$ is reducible to rational numbers, their subsets and so on in the same way as 2BP is reducible to rigid bodies and their interactions? We argue that the actual content and conceptual richness present in the concept of real numbers is not captured by the transfer of conceptual framework in this case. The question boils down to the following. Suppose C is reducible to B and A is isomorphic to C. Is A reducible to B as a concept? If the isomorphism could act as a reduction, this is indeed the case, because of transitivity of the "reducibility" relation.

However, there is no reason to regard an isomorphism as carrying a transfer of the conceptual network from real numbers to set theory, see Figure 1(Left). The isomorphism is an extra step which cannot be read out from the elements of B and their interactions. The difference between prime numbers and $\mathbb{R}$ is thus akin to the difference between the

Box 1: Dedekind construction of the reals

For a given set A, define its successor to be $s(A) = A \cup \{A\}$. It follows from the axiom of foundation that $A \neq s(A)$. Then start with the empty set $\varnothing$ and inductively take successors to obtain the sequence of finite von Neumann ordinals:

$$\varnothing,\ \ s(\varnothing) = \{\varnothing\},\ \ s(s(\varnothing)) = \{\varnothing, \{\varnothing\}\} \ldots$$

Denote the set of all finite von Neumann ordinals (which, formally, is the unique smallest set containing $\varnothing$ and closed under the successor operation) by $\mathbb{N}$ and call its elements natural numbers. On the meta-level we identify 0 with $\varnothing$, 1 with $\{\varnothing\}$ and so on. The existence of $\mathbb{N}$ is an axiom of ZFC, the so-called infinity axiom. Without the infinity axiom it is consistent that all sets are finite. Let us then define an operation on $\mathbb{N}$ which we denote by $+_\mathbb{N}$. Formally $+_\mathbb{N}$ is a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ there is a unique k with $(n, m, k) \in +_\mathbb{N}$. This can be denoted also by $n +_\mathbb{N} m = k$. $+_\mathbb{N}$ is defined by induction on s: $n +_\mathbb{N} 0 = n$ and if $n +_\mathbb{N} k$ is defined, then $n +_\mathbb{N} s(k) = s(n +_\mathbb{N} k)$.

Define an equivalence relation $E_\mathbb{Z}$ on $\mathbb{N}^2$ (formally a subset of $\mathbb{N}^2 \times \mathbb{N}^2 = \mathbb{N}^4$) by declaring that two pairs (n, m) and (p, q) are equivalent if and only if $n +_\mathbb{N} q = p +_\mathbb{N} m$. Define $\mathbb{Z}$ to be the set of equivalence classes of $E_\mathbb{Z}$. Denote the equivalence class of (n, m) by $[(n, m)]_\mathbb{Z}$ and define[a] the operation $+_\mathbb{Z}$ on this set by $[(n, m)]_\mathbb{Z} +_\mathbb{Z} [(n', m')]_\mathbb{Z} = [(n +_\mathbb{N} n', m +_\mathbb{N} m')]$. On the meta-level we identify the elements of $\mathbb{Z}$ with integers. For example $-1 = [(0, 1)]_\mathbb{Z}$, $0 = [(0, 0)]_\mathbb{Z}$, and $1 = [(1, 0)]_\mathbb{Z}$. Each integer is thus an infinite set of pairs of natural numbers.[b]

The rational numbers $\mathbb{Q}$ are defined from $\mathbb{Z}$ in the same way as $\mathbb{Z}$ is defined from $\mathbb{N}$ by using multiplication instead of addition. To do that, first define multiplication on $\mathbb{N}$ by induction and then define it on $\mathbb{Z}$ by[c] $[(n_0, n_1)]_\mathbb{Z} \cdot [(m_0, m_1)]_\mathbb{Z} = [(n_0m_0 + n_1m_1, n_0m_1 + n_1m_0)]_\mathbb{Z}$. Proceed to define $E_\mathbb{Q}$ to be the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ where (n, m) and (p, q) are equivalent[d] if $nq = pm$, and $\mathbb{Q}$ is the set of $E_\mathbb{Q}$-equivalence classes; each rational number is an infinite set of pairs of integers. Define operation $+_\mathbb{Q}$ on $\mathbb{Q}$ by $[(n, m)]_\mathbb{Q} +_\mathbb{Q} [(n', m')]_\mathbb{Q} = [(nm' +_\mathbb{Z} n'm, mm')]_\mathbb{Q}$ where $[(n, m)]_\mathbb{Q}$ denotes the $E_\mathbb{Q}$-class of (n, m). Define multiplication on $\mathbb{Q}$ by $[(n, m)]_\mathbb{Q} \cdot [(n', m')]_\mathbb{Q} = [(nn', mm')]_\mathbb{Q}$. Next we have to define an ordering on $\mathbb{Q}$ by defining one first by induction on $\mathbb{N}$, then use that to extend it to $\mathbb{Z}$ and use that to define it on $\mathbb{Q}$. We skip the details and assume that $<$ is defined on $\mathbb{Q}$ and coincides with the meta-level natural ordering of the rationals.

A *Dedekind cut (DC)* is a proper non-empty subset D of $\mathbb{Q}$ which has no $<$-greatest element and which is downward closed w.r.t. $<$, i.e.

$$\forall x, y \in \mathbb{Q}[(x \in D \,\&\, y < x) \to y \in D].$$

Denote the set of DC's by $\mathbb{D}$. We can then define the the relation $<_\mathbb{D}$ and operations $+_\mathbb{D}$ and $\cdot_\mathbb{D}$ on $\mathbb{D}$. The order $<_\mathbb{D}$ is just the subset relation: $A <_\mathbb{D} B \iff A \subsetneq B$ for all DC's $A, B \in \mathbb{D}$.

Given two DC's A and B, let $C = A +_\mathbb{D} B$ be the set of all rational numbers of the form $p +_\mathbb{Q} q$ where $p \in A$ and $q \in B$. It is standard to check that C is a DC.

To define $\cdot_\mathbb{D}$ is a little bit more tricky and we will not do it here in full detail. The main issue is that Dedekind cuts are based on the ordering $<$ of $\mathbb{Q}$ while multiplication by a negative number reverses $<$, so the naïve "$A \cdot B = \{pq \mid p \in A, q \in B\}$" does not work. To circumvent this, one first defines the operation $A \mapsto -A$, which corresponds to multiplying by (-1), then one defines the product AB for two *positive* cuts A and B (a cut is positive iff it contains 0) and finally one defines:

$$A \cdot_\mathbb{D} B = \begin{cases} AB, & \text{if } 0 \in A, 0 \in B, \\ (-A)B, & \text{if } 0 \notin A, 0 \in B, \\ A(-B), & \text{if } 0 \in A, 0 \notin B, \\ (-A)(-B), & \text{otherwise.} \end{cases}$$

The set $\mathbb{Q}$ can be identified with a subset of $\mathbb{D}$ through $q \mapsto \{p \in \mathbb{Q} \mid p < q\}$. In particular $0_\mathbb{D} = \{p \in \mathbb{Q} \mid p < 0\}$ and $1_\mathbb{D} = \{p \in \mathbb{Q} \mid p < 1\}$.

The model $(\mathbb{D}, 0_\mathbb{D}, 1_\mathbb{D}, <, +, \cdot)$ can now be shown to satisfy the second order axioms of the ordered field of real numbers.

[a] Here, and in similar places later, one should check that the definitions are independent on the choice of representatives of the $E_\mathbb{Z}$-classes.

[b] This is reminiscent of the idea of Frege (1884) and Russell (1903) where number 3 is the class of all triads in the world. In this case it is a way of coding subtraction into the system. Note that the defining equation $n + q = p + m$ is equivalent to $n - m = p - q$ which is the intention behind the definition of $E_\mathbb{Z}$.

[c] For notational convenience we do not use separate notations $\cdot_\mathbb{N}$ and $\cdot_\mathbb{Z}$ (nor later $\cdot_\mathbb{Q}$) to distinguish between the corresponding operations on $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ even though formally they are all different. Instead we write either $n \cdot m$ or simply nm.

[d] Now the intention is that $nq = pm$ is equivalent to $n/m = p/q$.

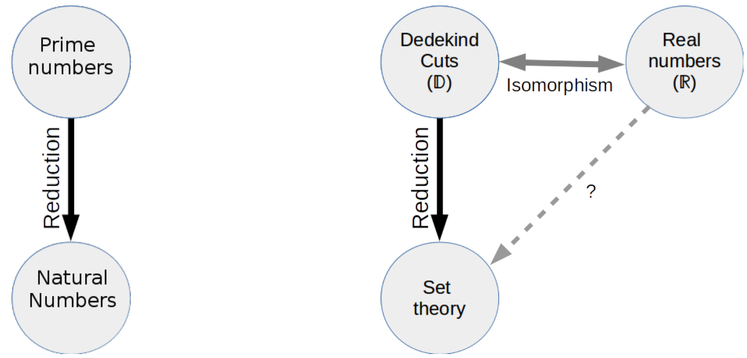


Figure 1: Left: Prime numbers are defined in terms of natural numbers and their operations. If the concept of $\mathbb{N}$ is acquired, then the definition of the prime nubmers is sufficient to grasp the concept. **Right**: The same can be said about Dedekind cuts and set theory. However, the concept of real numbers is not reducible to set theory. This is not the case even via the isomorphism, because the isomorphism does not provide a reduction of the conceptual framework. It has to be created separately.

two-body and three-body problem. In order to understand or describe the three-body problem, a new conceptual framework has to be introduced (one has to bring in the language of approximations, chaos theory, topology and so on). Similarly, the conceptual framework provided by set theory does not give sufficient grounds for recognising Dedekind cuts as real numbers. If we follow the conceptual transfer along the construction in Box 1, we will see that each real number is an *infinite set of infinite sets of pairs of infinite sets of pairs of finite von Neumann ordinals*. But clearly, when calling this "a real number" or "a point on a line", we are making a conceptual commitment which is nowhere to be found in, or explained by, set theory.

Thus, if we shared the conceptual understanding of set theory as well as the necessary first-order logic with an intelligent alien species and we tried to convey the idea of the real number line to them using only the Dedekind cut construction, we might fail. Presenting them with this construction would be like showing a compiled binary file of a chess-playing program to someone unfamiliar with chess, but who knows what a binary file is. Based on that file, we would not expect that person to figure out what the game of chess is about.

We can see that this disctinction between different conceptual frameworks is due to taking an epistemological approach to mathematical definitions. Had we taken the ontological approach, it would be feasible that $\mathbb{R}$ is reducible to sets. To be more precise, this would be the case if we accepted an ontological version of *structuralism*. According to structuralism, mathematical ontology does not concern mathematical objects (such as numbers or sets) but rather mathematical structures (such as Peano Arithmetic or ZFC set theory) (Shapiro 1997). Under the structuralist interpretation, if A is isomorphic to some system C that is emergent from B, A and C are the *same* system, and thus also A would have to be emergent from B. With an ontological understanding of structuralism, there would be no difference between the systems A and B. But with the epistemological approach, this changes. B alone does not provide the conceptual framework for A, and as such A is not reducible.

It is this epistemological approach that we are concerned with in this paper, and in Section 5 we will argue that the epistemological angle is indeed central when we consider set theory as the foundation of mathematics. From what we have seen so far, summarised in Table 2, it is clear that conceptual transfer is an important question for mathematics, just as it was for physics. In the ontological approach, however, none of these distinctions could be made.

4 Descriptions and Constructions

Some set theorists would argue that the purpose of constructing the system $\mathbb{D}$ of Dedekind cuts is not to reduce $\mathbb{R}$ to set theory, but to show that the existence of a complete ordered field is a consequence of ZFC and it is therefore consistent if only ZFC is consistent. But even here one has to be careful: a complete ordered field is not something that can be defined in the language $\{\in\}$ of set theory, one needs the vocabulary and the language $\{+, \cdot, <\}$ of ordered fields. So the talk about the field is either done on the meta-level, or one defines a formalism *within* set theory in which case the vocabulary, language and formulas of ordered fields are represented by sets. In the former case, the meta-level, not set theory, is the source of the conceptual framework. In the latter case, the conceptual framework of talking about a formalism is brought from outside set theory. But even granting this, we have the following situation: there is a formula $\varphi_D(x)$ saying that x is the system $\mathbb{D}$ of Dedekind cuts of rational numbers together with the operations as defined in Box 1, and a formula $\varphi_F(x)$ saying that x is a tuple satisfying the axioms of a complete ordered field in the within-ZFC-formalism. So it seems that we have two formulas here which, on an intuitive level, talk about the same thing.

Why would anyone need two different formulas for the same thing? The answer is that they fulfil two different functions. The purpose of $\varphi_D(x)$ is that it is easy to prove (from ZFC) that $\exists x \varphi_D(x)$. The purpose of $\varphi_F(x)$ is to conform to our intuition of what the real number line "should really be like". This culminates of course in the fact that we can also prove that

$$\forall x(\varphi_D(x) \to \varphi_F(x)) \tag{1}$$

i.e. that the system $\mathbb{D}$ is a complete ordered field. The sentence (1) implies another, even more comfortable-looking sentence

$$\exists x \varphi_D(x) \to \exists x \varphi_F(x)$$

which completes the proof that the object of our intention can be "found" in set theory. Thus, there was, all along, an idea, an intuition *behind* this process which is embodied in φ_F.

Philosophically, this situation leads us to a distinction between two types of mathematical definitions which we call *descriptions* and *constructions*.[3] In philosophy of mathematics, many approaches (e.g. Nagel, 1935; Shapiro, 1997) take constructions to be one

[3]We do not intend to refer to the constructivist approaches in the philosophy of mathematics where the only valid constructions are those that do not rely on the law of excluded middle or the axiom of choice (Troelstra & van Dalen, 1988).

method of introducing new entities to mathematics (alongside stipulations and implicit definitions). Construction is thus seen as a particular type of a definition, one in which new entities are defined as combinations of previously defined objects. However, from the epistemological point of view, such constructions can be divided into two classes: what should be called *proper* constructions and descriptions. This distinction is important for our discussion on concept acquisition because the above situation seems to be a typical one: there are multiple definitions which aim at capturing the same mathematical idea. Some of them reflect the conceptual framework of the defined object. Others capture formal properties of that object but as long as the reductive transfer of the conceptual framework is the only way in which they can be understood, they can be conceptually totally different.

To give another example, the definition of π in the form "the ratio of the circumference of a circle to its diameter" is a description while the (mathematically equivalent) definition

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} \tag{2}$$

is a construction.

Both constructions and descriptions are definitions that introduce new concepts to mathematics, but they do so in fundamentally different ways. We propose that difference is best understood via transfer of conceptual framework. Constructions are use to introduce concepts that are not reducible to the basic concepts used in the construction process (e.g., sets vs. real line). In descriptions, however, the conceptual framework transfers (e.g., natural numbers vs. primes).

To sum up, epistemologically speaking, whatever is created from rational numbers by taking their Dedekind cuts is not the real number line. As a definition, it is a construction and not a description. Dedekind cuts are a mathematical object which can be shown to be isomorphic to real numbers. But if it is thought of as *the* reals, the conceptual framework must have been attached to it from somewhere else.

5 Problems of Set Theory as a Foundation

The problem of thinking of real numbers as Dedekind cuts is particularly pertinent because these kinds of constructions are at the basis of foundationalism in mathematics. It is often believed that higher-level mathematical concepts should be defined in terms of lower-level ones, and in this hierarchy the foundation is usually thought to be provided by sets.

Maddy (2017) isolates several ways in which set theory – the Zermelo-Fraenkel axiom system with the Axiom of Choice (ZFC), and set theoretic foundations in general – is perceived as a foundation. These include: *Meta-mathematical Corral* which means that set theory provides means to investigate provability and consistency claims made in mathematics, i.e., essentially as a tool to study the consistency hierarchy; *Epistemic Source* according to which we can use set theory to "check" whether some mathematical statement holds or not by checking whether it is provable in ZFC; *Metaphysical Insight* meaning that "the set-theoretic reduction of a given mathematical object to a given set actually reveals the true metaphysical identity that object enjoyed all along"; *Elucidation* which means that a set-theoretic replacement of an imprecise notion makes it precise; and finally *Shared Standard*, meaning that "if you want to know whether or not a so-and-so exists, see whether one can be found in [the universe of sets]; if you want to know whether or not such-and-such is provable, see whether it can be derived from the axioms of set theory." In the latter Maddy does not mean ontological, Platonic, existence, but rather a means of reaching practical agreement between mathematicians.

Maddy points out that even though we may agree on the need for set-theoretical foundations, we may disagree on the degree to which set theory acts as an embodiment for all of mathematics. Some say that mathematical objects *are* sets and others say that they can merely be *reduced* to sets (see quotes below).

In this paper we have argued that many mathematical concepts cannot be reduced to set theory in an epistemologically feasible way. Thus, while we cannot rule out the possibility of *Metaphysical Insight*, it is epistemologically unfounded. Since the conceptual framework of many mathematical objects cannot be read out from their set-theoretic definitions, these definitions cannot be feasibly thought as giving epistemic access to those concepts.

Similar problems arise with the other uses Maddy sees for set theoretical foundations. In both *Shared Standard* and *Epistemic Source*, it is not clear what it means to "find something in the universe of sets". Under our approach, it cannot refer to merely finding some set-theoretic construct which is *isomorphic* to the object, which takes away much of the power of set theoretic foundations.

Of course we do not mean to suggest that set theory does not have any of the benefits associated with the above uses. Instead, we want to point out that taking the epistemological approach to set theoretic constructions, i.e, taking the question of concept acquisition seriously, some of the uses are compromised.

Here is how we see the standard foundationalist story of concept formation unravel. First it is claimed that (A) the concepts of "set" and "inclusion" are very easy to explain and to back up philosophically. They are conceptually simple and it is possible to establish simple truths about them by which a recursive set of meaningful axioms can be motivated. Second (B) almost all mathematics can be defined in, and deduced from ZFC. From (A) and (B), it is often concluded that (C) ZFC is a solid foundation which in particular enjoys *Metaphysical Insight*. A round-up of set theory textbooks shows how prevalent these views are.

Argument (A) can be found, for example, in the following passages:

> [One of the primary aims] is to explain systematically what the most basic and general objects of mathematics [sets] really are and why they behave as they do. (Eisenberg, 1971)

> [The pairing and union axioms of ZFC] are nearly too obvious to deserve comment from most commentators. When justifications are given, they are based on ... vague intuitions about the nature of sets. (Maddy, 1988)

This latter quote from Maddy shows what the status of set theoretic axioms is generally seen to be: almost too obvious to deserve comments. Equally ubiquitous is the support for (B):

> Working within ZFC, one develops: [...] All the mathematics found in basic texts on analysis, topology, algebra, etc. (Kunen, 1980)

> Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate ... axioms about these primitive notions ... From such axioms, all known mathematics may be derived. (Kunen, 1980)

> Indeed, all mathematical entities can be represented as sets. (Eisenberg, 1971)

> All branches of mathematics are developed, consciously or unconsciously, in set theory or in some part of it. .... We shall show how the concepts of ordered pair, relation and function, which are so basic in mathematics, can be developed within set theory.[4] (Levy, 2012).

> Among the many branches of modern mathematics set theory occupies a unique place: with a few rare exceptions the entities which are studied and analysed in mathematics may be regarded as certain particular sets or classes of objects.[5] (Suppes, 1960)

The following passage supporting (B) is particularly interesting in that it explicitly dismisses the role of geometric intuition once the foundationalist reduction to ZFC is done:

> [about the Dedekind-construction of the reals:] The arithmetisation of analysis, brought about by Dedekind, Weierstrass, and others, succeeded in developing an algebraically self-contained notion of real number without any appeal to geometric intuition. (Hatcher & Bunge, 2014 (first edition 1982))

In this paper we argue that this is kind of argument is strictly speaking false: if the arithmetisation was really done "without any appeal to geometric intuition", then we wouldn't recognise these objects as real numbers. This does not mean that we do not see an important role for set theory for the foundations of mathematics. But we do claim that the usefulness of set theory as a foundation should be tied to the conceptual framework of set theory. In this respect, we are inclined to agree with the more moderate position of Jech:

> The axioms of ZFC are generally accepted as a correct formalisation of those principles that mathematicians apply *when dealing with sets*. [Our emphasis] (Jech, 2003)

[4] Note that the set theoretic universe can be represented as a graph, but nobody is claiming that we are unconsciously doing graph theory when doing functional analysis. (Our footnote)

[5] We are curious to know what are the "rare exceptions" to Suppes. He doesn't seem to give any in the book.

Generally speaking, such voices of dissent are in the minority, and set theory is seen as a tractable and solid foundation for mathematics, often implicitly including the assumption of self-contained conceptual and semantic treatment. Part (A) of this reasoning we are willing to accept. Sets are indeed very easy to understand, as well as being philosophically simple – or at least simpler than many other mathematical notions. There are also plausible accounts on how the intuitions, conceptual frameworks and semantics of sets could be grounded in human conceptual system (Lakoff & Núñez, 2000; Pantsar, 2015). As far as the argument (B) is considered, however, we claim that the reasoning is flawed. Even granting that sets and inclusions are conceptually understood, and granted that we can derive simple truths about them – such as featured in the ZFC axioms – we still lack a conceptual foundation for other mathematics.

So far our argumentation has been mostly negative in nature. We have tried to show that in order to understand the concept of the real number line, it is not sufficient to understand what a set is and what inclusion is. Next we will try to add a positive angle to this argument by proposing a framework for how conceptual frameworks can emerge and concepts acquired in mathematics in non-reducible ways.

6 How are mathematical concepts obtained?

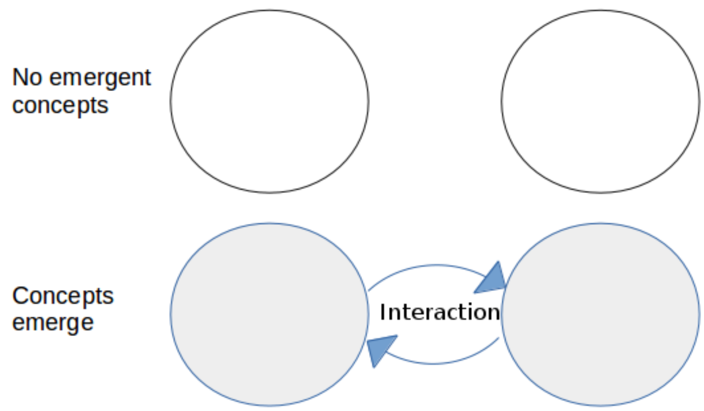
> If we are to understand embodied cognition as a natural consequence of rich and continuous recurrent interactions among neural subsystems, then building interactivity into models of cognition should have embodiment fall out of the simulation naturally.
>
> (Pezzulo et al., 2011, p. 16).

Above we have argued that the view that set theory provides intuitive and tractable foundation for mathematics is quite problematic. While we can carry out the set-theoretic reconstruction of objects such as real numbers, this process is only possible by bringing in concepts from other conceptual frameworks. Thus, there is something essential to the concept of real numbers that cannot be derived from the lower-level concepts.

But how *do* we grasp mathematical concepts such as real numbers, if not in terms of lower-level concepts? What makes us see Dedekind cuts as real numbers? Indeed, how do we generally acquire mathematical concepts?

It should be noted that this question of concept acquisition and semantics of mathematical expressions is central to all epistemological accounts of mathematics. Connecting mathematical symbols to their abstract meanings by cognitive agents needs to be explained even in the Platonic view where mathematical reality exists independently of the agent. There are many empirical accounts on how concepts can be grasped by a cognitive system and how symbols are connected to their meanings as well as wide a variety of theories on mathematical understanding. Here we will isolate a common pattern between a representative set of these accounts. We propose that this common aspect of these theories should be explored in more detail to understand the acquisition of abstract concepts in general, and mathematical ones in particular.

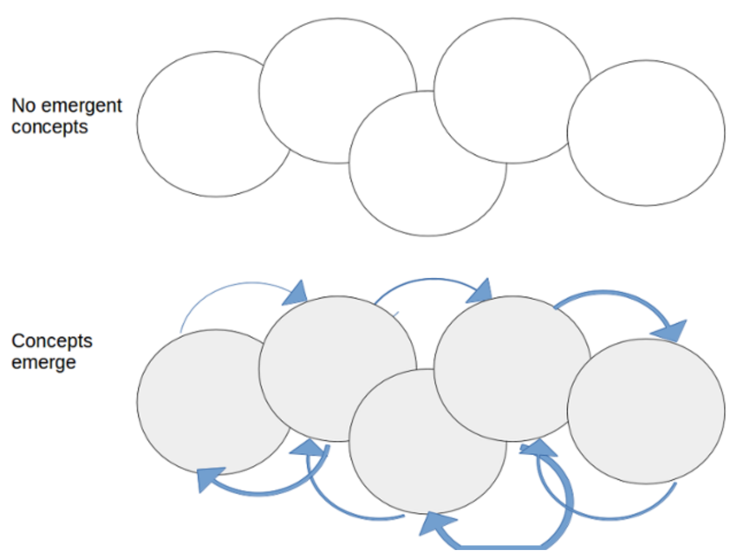
The common pattern, in fact, is already familiar to us from the context of emergence and reduction: new qualities emerge when there is non-trivial interaction between the simpler components. The emergence of new conceptual frameworks follows an analogous basic structure to how emergent properties arise:



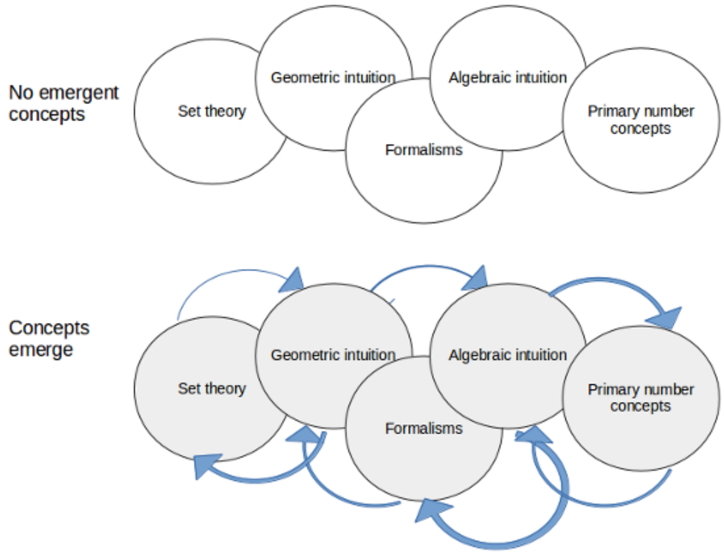
When we have no gravity between bodies, there are no dynamics; when there is gravity, dynamics emerge. Sometimes, like in the case of two-body problem, these dynamics can be explained with the same theory that was used to explain the bodies. However, in cases like the three-body problem, the phenomenon is epistemologically emergent and new modes of explanation are needed. Conceptual understanding must be drawn from a further source.

Analogously, set theoretic formalism alone does not constitute real numbers and neither does geometric intuition. However, their combinations and interactions in a skillful activity give rise to something closer to what could be called real numbers. We will now turn to several examples drawn from literature on cognitive science, as well as philosophical and historical accounts of mathematical concept creation, to support and clarify this view.

Of course the picture above is very simplistic and the reader should keep in mind that in each example below, an accurate image would look more like this:

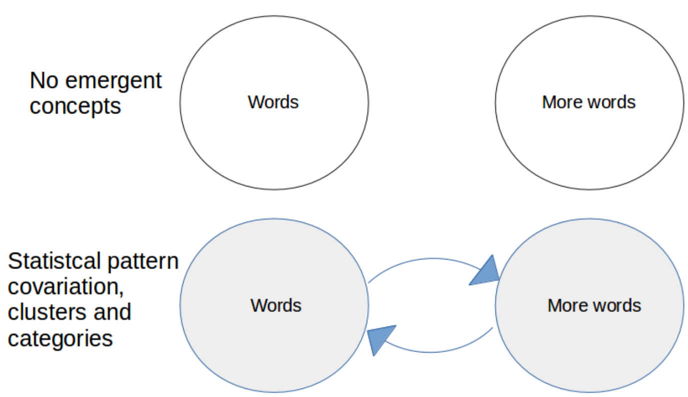


which in the case of abstract mathematical concepts could look something like this:



6.1 Statistical co-variations

Landauer and Dumais (1997) proposed latent semantic analysis (LSA) as a model for how humans acquire, use and recognise meanings. LSA is a statistical method in which a word's meaning arises from the way words are connected to other words through co-occurrence patterns. Landauer and Dumais (1997) claim that the resulting algorithm is comparable with children's performance. More recently, statistics and methods based on machine learning have achieved a totally new level of performance (Mikolov, Sutskever, Chen, Corrado, & Dean, 2013). In this view the meaning of one word depends on its interaction with other words in the vocabulary. For example the meaning of the words "elephant" and "New York" would be discriminated through their co-occurrence patterns with words like "animal", "city", "eyes", "sky scrapers", "ivory". Here meaning emerges from the interaction between symbols:



6.2 Symbol grounding problem and sensorimotor contingencies

However, even though a machine learning algorithm can achieve and beat human level performance in word pattern matching through statistical co-variation, it is no proof that machines understand what they are doing, nor is it explained why connecting words to other words would make them any more meaningful. Inferring meaning only from patterns of words may miss their reference. This can be seen from the symbol grounding problem, as originally formulated by Harnad (1990). How is it ever possible to escape the circle of "symbol merry go around"? For example, here are the two words ("elephant" and "New York") in Thai:

นิวยอร์ก
คชสาร

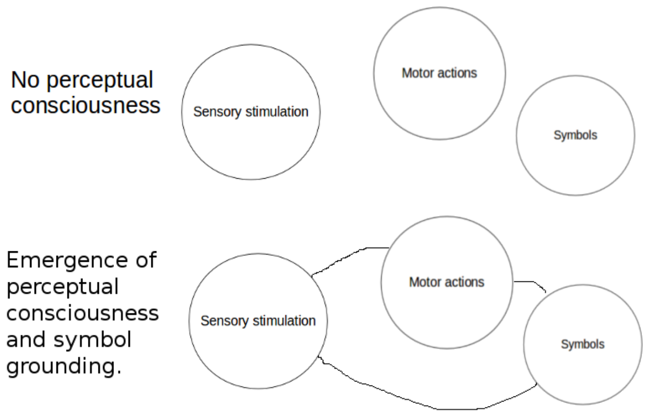
If you are unfamiliar with Thai, there is little chance you can tell which one is which, and it won't help if one points to other Thai words to which these are associated. Thus, mere symbols don't seem to be enough. Harnad compared this to someone with zero knowledge in Chinese trying to learn Chinese from a Chinese-Chinese dictionary alone. Similar critique has been issued against LSA as a model of human semantic content acquisition (Barsalou, 1999, 2008). In our terminology, this problem can be understood through the non-transfer of the conceptual framework. Patterns of words may not be sufficient to establish the conceptual framework in which the words are used.

A solution many authors propose is embodiment and grounding meaning in the sensorimotor domain (Barsalou, 1999, 2008; Noë, 2004). According to this view, meaning of words is grounded in sensorimotor interactions with the environment. This theory has received support from neuroscience (Hauk, Johnsrude, & Pulvermüller, 2004), but the grounding process merely seems to push the problem further. Cubek, Ertel, and Palm (2015) argue that even the embodied approach does not solve the problem of semantic commitment. This problem can be seen more clearly if we consider a robot which operates at its base with 0's and 1's – as all computers today do.[6] Then, in the end of the day, all the sensorimotor interactions are manifested as patterns of these symbols (0's and 1's) and no matter how the "symbol grounding" is implemented, it is likely to be a coupling between some "higher level" symbols and these "lower level" symbols. Again, we are essentially back down to symbol co-variation.

On the other hand, O'Regan and Noë (2001); Noë (2004) put forward an account according to which perception is not a passive, but an active process and that perceptual consciousness can only arise in a dynamic and skillful coupling between actions and sensations called sensorimotor contingencies. According to O'Regan and Noë (2001), neither pure sensations nor pure motorics constitute perception, but it is achieved through their coupling. A neuroscientific approach has recently been proposed by Ahissar and Assa

[6] An exception is created by quantum computers, but they are also operating with pieces of information (qubits) that bear no direct connection to things they are thought to represent, if anything.

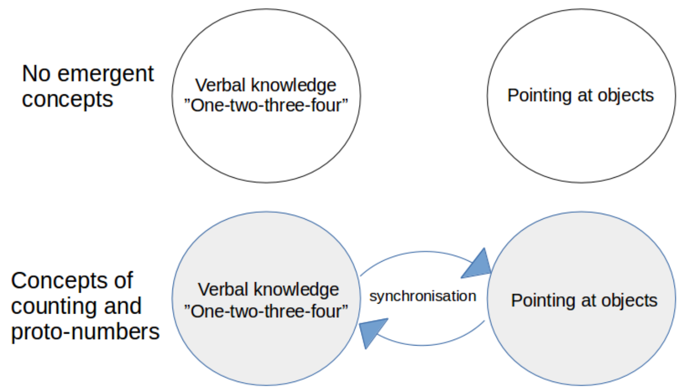
(2016). These two views combined suggest the following picture:



6.3 Learning to count

When we discuss the emergence of conceptual frameworks, one interesting case is the way children acquire natural number concepts. Initially, children only learn to recite the numeral sequence, without being able to match it to quantities. At that stage, a child can be familiar with the verbal recitation "one-two-three-four-five", yet she cannot use the numerals to refer to quantities (Fuson, 2012; Wynn, 1990; Davidson, Eng, & Barner, 2012). In this way, children can acquire a syntactic placeholder structure without positing any semantic content to its members. It is only later that they gradually start grasping that members of the numeral sequence can be used to refer to discrete quantities.

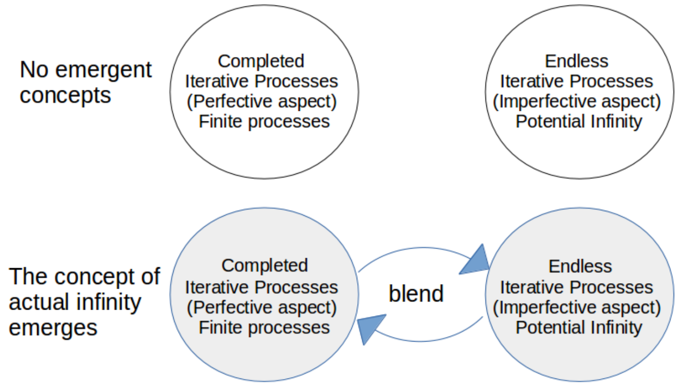
A popular explanation of how this happens is the process of "bootstrapping" (Sarnecka & Carey, 2008; Carey, 2009; Spelke, 2011). Taking learning in a holistic Quinean (1960) context, the bootstrapping theory proposes that from the existing web of representations that children have from processing quantitative information, semantic content emerges that enables saturating the placeholder structure with meaning. More specifically, starting from familiarity with a small group of numerals, children "bootstrap" an early form of the successor function, thus making them grasp the progression of natural numbers. An important stage in this is acquiring what (Sarnecka & Carey, 2008) call the "cardinality principle", i.e., the understanding that the numeral list can be continued indefinitely and every numeral on the list refers to a unique discrete quantity.



How this process exactly happens is still largely a matter of conjecture, but the bootstrapping theory provides a fruitful framework for explaining how our pre-verbal ability with numerosities (see e.g. Dehaene, 2011) develops into arithmetical knowledge (Carey, 2009). This process does not happen automatically, as seen in the various non-arithmetical cultures, such as the Pirahã and the Munduruku of the Amazon (Gordon, 2004; Pica, Lemer, Izard, & Dehaene, 2004) The best current hypothesis is that a sufficient amount of cultural and linguistic input (starting from the existence of a rich enough numeral system) combined with the early proto-arithmetical ability with numerosities together make it possible to grasp the number concepts (see also Núñez, 2017). What we know is that the process is not a simple case of children learning a numeral and picking out its reference. In some much more holistic manner, concepts are grasped on top of an initially acquired syntactic structure without the proper semantic content.

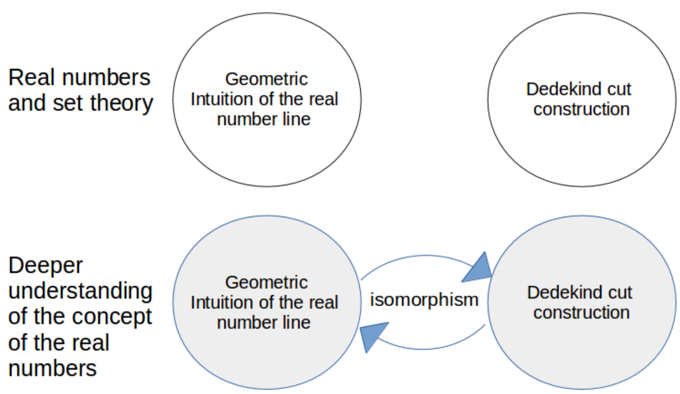
6.4 Metaphor blending

Expanding on the origins of number concepts, Lakoff and Núñez (2000) have proposed that the understanding of mathematical concepts is grounded in embodied representations, and the conceptual frameworks are transferred via metaphor maps. In terms of conceptual metaphor theory (Lakoff & Johnson, 1980), these maps transfer concepts and structure from a source domain (that we are familiar with) to a target domain (that we are trying to explain). In this way, for example, the experience of comparing lengths or distances (source domain) is metaphorically mapped onto the real line (target domain), which enables us to understand the relations "$<$" and "$>$" in mathematics. An interesting concept in this theory is the concept of a metaphor blend. This is where two "source domains" are blended to yield a new, possibly more abstract, target domain. An example is what Lakoff and Núñez call the *basic metaphor of infinity* (BMI) (Núñez, 2005). The idea in BMI is that the notions of perfective and imperfective aspects get metaphorically mapped to the same, new, domain yielding the idea of actual infinity. For instance the verb *to jump* includes the idea of an ending as in "I jumped", but *to run* does not imply an ending as in "I ran". Metaphorically blending these two notions leads to a new conceptual framework where an activity is ongoing, but has a final state nonetheless.



Pantsar (2015) has presented an alternative metaphorical approach to mathematical concepts, with the focus on *processes*. In this account, mathematical concepts (or objects) are seen as metaphorical counterparts of mathematical processes, which in turn have their basis in processes carried out in our physical environment. Natural numbers, for example, enter mathematical discourse as objects because they are the target domain for the metaphorical discourse of processes as objects, based on the source domain of the process of counting. Counting, in turn, is a process tightly connected in its origin to physical processes, such as, extending fingers or tallying.

Concerning real numbers and getting back to the Dedekind construction, one could argue that the concepts of the real number and the Dedekind cut are *cognitively* different. We may use different metaphors and different processes between understanding the two, thus having different conceptual frameworks. The realisation of the isomorphism between them is yet another cognitive achievement which ties them together into a more grounded concept:



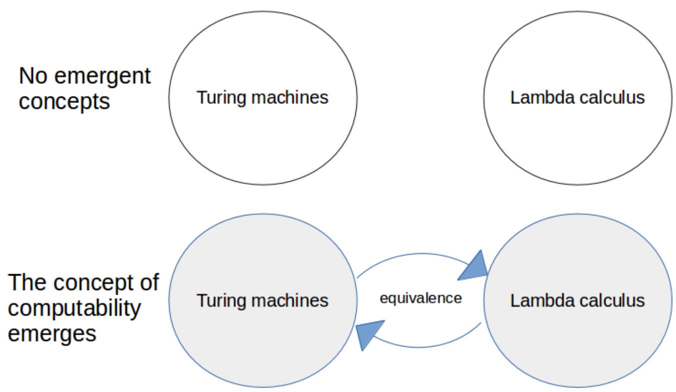
6.5 Computable functions

This multifaceted use of related concepts and emerging frameworks can be seen in the history of mathematics. For example, let us consider a development in mathematical logic in which a new concept emerged in mathematics based on considerations of decidability.

Based on the logicist project of Frege, and Whitehead, Hilbert formulated his dream of formalising the whole of mathematics. The main theoretical component of that dream was the search for a formal procedure for being able to decide of every mathematical statement whether it is true or false. This inspired Gödel (1934) to formally define what "formal" means and prove that their dream was untenable. The parallel development of the science of computation led Turing (1937) to define computation so as to prove the undecidability of certain statements which further strengthened the robustness of Gödel's result. Finally, Church (1936) also gave a model of computation and recursive functions to show the undecidability of certain problems in number theory. To quote Blum (1990):

> It is quite remarkable that even though the formalisms were often markedly different, in each case, the resulting class of computable functions (and hence decidable sets) was exactly the same. Thus, the class of computable functions appears to be a natural class, independent of any specific model of computation. (Blum, 1990)

Thus, out of a variety of formalisms, because of their equivalence, a natural concept had emerged:



In what has become known as the Church-Turing thesis, this shared idea of computability came to characterise how mechanical procedures in mathematics were understood. Furthermore, the class of computable functions also came to be equated with the class of functions solvable by computers.

6.6 Symmetry

It is not always the case that different frameworks provide such a remarkable degree of coherence and equivalence as was the case with computation. The notion of symmetry, for example, is less robust, but the concept is nonetheless strengthened by the fact that there are mathematical models and non-mathematical descriptions of symmetry which are coherent with each other to some degree. Group theory is of course almost a prototype of what we understand as symmetry. However, we do not want restrict ourselves to groups. Sometimes groupoids, monoids and quandles are better at describing some symmetries.

On the one hand, in art, symmetry may appear in contexts where the approach of mathematical group theory is inapplicable; for example an artist may want to emphasise some symmetries by breaking others. On the other hand, things that can be described by groups do not always, at first glance, have anything to do with symmetry. Take for example the fundamental group or the group the permutations of the Rubik's cube. However, in most cases, things that obey the laws of group theory, do also satisfy our intuitions (in other frameworks) of what symmetry is.

6.7 Emergence of Conceptual Frameworks in Mathematics

We outlined several mechanisms studied in the literature which may account for creation of new conceptual frameworks and we hypothesise that this is how it is happening in mathematics as well. When we first learn about the number line, we already have a lot of different conceptual background about numbers from subitizing to simple addition, finger-counting, etc. The geometric intuition about the number line is superimposed on these prior intuitions and through the right type of an interaction, our conception of the number line becomes more robust. If we go on to study set theory and learn about Dedekind cuts, we are given an opportunity to connect our existing conceptual network to the newly learned construction and make our concepts even richer and more elaborate, by combining conceptual frameworks.

Parallels to the present discussion can be found in the famous distinction between intension and extension by (Frege, 1948). Just like Frege distinguished between Phosphorus (Morning Star) and Hesperus (Evening Star) as two intensions of the same reference (the planer Venus), we can see different conceptual frameworks as giving different intension for mathematical objects. In order to establish that the concepts "Phosphorus" and "Hesperus" in fact have the same extension, more was needed than merely looking at sky in the morning and in the evening. By making observations, constructing theories and testing predictions, it was possible to establish that "Phosphorus" and "Hesperus" refer to the same celestial body.

In our view, a similar thing happens with mathematical definitions when conceptual frameworks are combined. Dedekind cuts are set theoretical concepts, but by showing them to be isomorphic with real numbers, we can show that the Dedekind cuts and real numbers have the same extension. Importantly, this process is not possible from the purely set theoretical conceptual framework. Just like in establishing that "Phosphorus" and "Hesperus" have the same extension, a new richer conceptual framework is needed.

We compared this process to the way concepts can be acquired through the sensorimotor and embodied domain to emphasise that the mechanism of concept acquisition can be similar on different levels of abstraction. The way we are searching for affordances in order to conceptualise the space around us, we are looking for possible ways of manipulating our mental images of formulas and constructions, as well as physical representations such as symbols and diagrams, in order to find coherence and conceptualise the world of mathematical objects.

7 Conclusion

We started by considering reducibility and emergence in the epistemological sense and argued that these notions can analogously be applied to analyse mathematical concepts. We then justified this analogy through the concept of transfer of conceptual framework. When a mathematical definition allows the conceptual transfer, the newly defined notion is reducible to the simpler concepts, otherwise it is not. Based on this, we argued that a central construction in set theory, the construction of the real number line through Dedekind cuts, is an example of a construction with no conceptual transfer. We thus concluded that the concept of the real number line is not reducible to set theory.

But if real numbers are not reducible to set theory, where do they come from? To answer this question, we outlined a theory of concept acquisition which we believe to be promising in understanding this issue in mathematics. This theory uses the idea of *emergence from interaction* as a part of how we come to grasp a concept. Analogously to complicated dynamics emerging in the three-body problem from the interaction (through gravity) of the bodies, concepts and conceptual frameworks would emerge as a result of interaction of various domains, frameworks or modalities such as different formalisms, geometric intuitions, embodied intuitions, social activities and so on the interactions being e.g. in the form of metaphor maps. Grounding concepts to these modalities through their interactions would be analogous to grounding more basic concepts to sensorimotor activities and multisensory integration.

The account presented in this paper is obviously meant to be a philosophical treatment of the topic, rather than suggesting an actual model of concept acquisition. However, already on this theoretical level we can see great potential in this dynamic model of the emergence of conceptual frameworks. One of the most important topics in the philosophy of mathematics is reconciling mathematics as a formal discipline on the one hand, and as a human activity with a great amount of applications on the other hand. If we see mathematics ultimately purely as symbol manipulation, this connection is difficult to explain. The present theory, however, has no such problems. Formal theories constitute one dimension of the multifaced nature of mathematical concepts, but so do all the outer frameworks that contribute to mathematical concept acquisition. This way, we hope that our approach can serve two purposes. First, it will allow generalising the existing approaches to symbol grounding, concept formation and meaning in a way that enables one to handle abstract concepts. Second, it can provide a philosophical background for the way mathematical concepts and conceptual frameworks can be grasped.

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