Non-reducible Conceptual Frameworks in Mathematics

Anonymised

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Abstract

In this paper, we identify an important explanatory role that an ideal foundational theory of mathematics ought to have. This role is to provide an explanation of the emergence of semantic and conceptual contents. To justify the need of such an explanation we develop a theory of reducibility and emergence of conceptual frameworks in mathematics and argue that many of them are mutually irreducible. We also argue that set theory as a foundational discipline does not provide such explanations. We then describe on a philosophical level what such a theory of emergence of conceptual content might look like. Finally we draw parallels to some theories in cognitive science.

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Sijaintisi

It's more or less standard orthodoxy these days that set theory – ZFC, extended by large cardinals – provides a foundation for classical mathematics. Oddly enough, it's less clear what 'providing a foundation' comes to.

Maddy (2017)

1 Introduction

What is the content of the following expressions: "real number line", "prime number" and "a basis of a separable Hilbert space"? The mathematical answer is

$$\textit{look at the definitions.} \tag{1}$$

While widely accepted in mathematics, this answer creates the problem of infinite regress, because the notions used in the definitions have to be defined using other notions and so on, ad infinitum. To overcome this problem, foundational disciplines, such as set theory, postulate certain undefined basic concepts and axioms. These basic concepts can be sets or classes. The Zermelo-Fraenkel set theory with the axiom of choice (ZFC) is perhaps the best-known example of a set of axioms that has been used as a foundation for mathematics.

The answer (1) assumes that the definition of a mathematical notion captures entirely its meaning, conceptual content and is sufficient to both understand and communicate it. But since infinite regress cannot take place, the meaning and content of the very basic notions, such as sets and classes, has to be understood (and communicated) in some other ways than by a mathematical definition. This of course necessitates that these "other ways" *exist*. So in contrast to (1), we have two possible ways in which mathematical conceptual content can be achieved: through mathematical definitions or without them, due to some other way. Based on this distinction, we pose the following question:

Question 1. *Can some mathematical notions be dependent on both?*

An affirmative answer to this question would entail that there is a mathematical notion d which is formally defined using some concepts $c_1, \ldots, c_n$, but the meaning, conceptual content, the understanding and communication of d still requires something else.

Let us use (R) to refer to the situation in which a mathematical notion and its conceptual content can be entirely captured, understood, and communicated solely by using a formal definition which refers to other, previously understood concepts. Here "R" stands for "reducibility". Let us refer by (E) to the situation when this is not the case. "E" stands for "emergence". We will see later why these labels are accurate.

Using this distinction we can define what we call *conceptual frameworks* (CF). Roughly speaking, a conceptual framework refers to a collection of ideas and concepts which is closed under reducibility (R). For example, the concepts of "set theoretic union" and "power set" belong to the same CF, that of set theory. Question 1 above can now be reformulated as follows:

Question 2. *Are there mutually irreducible CF's in mathematics? In particular, are there other CF's than that to which set theory belongs?*

In Section 3 we will give arguments in favour of a positive answer to Questions 1 and 2. But this prompts the philosophical question what the "undefined" notions that are not reducible to the CF are. The question "What are sets?" cannot be answered by (1), so another account is needed to explain where the notion of set comes from. However, this is only a particular case of a wider problem. If sets were the only undefined notions (along with perhaps classes and the membership relation), the task at hand would have a somewhat limited scope. This is often dismissed under the pretext of triviality ("sets are so intuitive and clear anyway"). But if, as we will argue, mathematics is abundant with mutually irreducible CF's, the issue becomes integral to the meaning of mathematical notions.

Thus, if the answer to Questions 1 and 2 is positive, it makes it even more pressing to answer the following problem:

Question 3. *What are the "other" ways of creating mathematical content? How does the existence of such ways square with the claim that "all mathematics can be defined in set theory?" How could a foundational theory fully account and explain this phenomenon?*

To answer Questions 3, in this paper we apply to mathematics the philosophical discussion on reducibility and (weak or epistemological Bedau (1997); Clark (1998)) emergence. We will see that there are many ways in which the situations (R) and (E) are analogous to reducibility and emergence. This is explored in Sections 5 and 6.

But to answer Question 3 in any satisfactory manner, it is not sufficient to simply state that what is not reducible, must be emergent. Reducibility and emergence are used here because they provide a fruitful framework for analysing the problem. But ultimately, the issue we must tackle is the following:

Question 4. *How exactly is mathematical content emergent? Are there systematic principles governing such emergence?*

To answer these questions, we develop a philosophical proposal of how such emergence could take place in the context of mathematical notions. Drawing from the Fregean distinction between intension and extension, we compare the process of mathematical meaning creation to the process through which the seemingly independent notions of "Morning star" and "Evening star" can be merged into one concept "Venus". To show this, we use an analogy from physical systems: the way complex interactions of physical bodies can create epistemologically new types of dynamics is analogous to the way in which complex interactions between existing conceptual frameworks can give rise to new notions. In the last part of this paper, we compare our approach to existing theories in the literature on the philosophy and cognitive science of mathematical practice.

2 Reducibility in set theory

It is often accepted that sets play a special foundational role in mathematics. A prevalent argument for that role in set theory text-books goes as follows:

(A) The concepts of "set" and "inclusion" are very easy to explain and to back up philosophically. They are conceptually simple and it is possible to establish simple truths about them by which a recursive set of meaningful axioms can be motivated.

(B) Almost all mathematics can be defined in, and deduced from ZFC.

(C) Since everything can be reduced to sets (B) and sets are metaphysically and conceptually well understood (A), so is everything else well understood too. It also follows that to know truths about mathematical objects reduces to knowing truths about sets.

The argument (A) is widely supported, as seen in the following passages:

> "[One of the primary aims] is to explain systematically what the most basic and general objects of mathematics [sets] really are and why they behave as they do." Eisenberg (1971)

> "[The pairing and union axioms of ZFC] are nearly too obvious to deserve comment from most commentators. When justifications are given, they are based on ... vague intuitions about the nature of sets." Maddy (1988)

This latter quote from Maddy shows what the status of set theoretic axioms is generally seen to be: almost too obvious to deserve comments. The support for (B) is ubiquitous:

> "Working within ZFC, one develops: [...] All the mathematics found in basic texts on analysis, topology, algebra, etc." Kunen (1980)

> "Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate ... axioms about these primitive notions ... From such axioms, all known mathematics may be derived." Kunen (1980)

"Indeed, all mathematical entities can be represented as sets." Eisenberg (1971)

"All branches of mathematics are developed, consciously or unconsciously, in set theory or in some part of it. .... We shall show how the concepts of ordered pair, relation and function, which are so basic in mathematics, can be developed within set theory." Levy (2012).

"Among the many branches of modern mathematics set theory occupies a unique place: with a few rare exceptions the entities which are studied and analysed in mathematics may be regarded as certain particular sets or classes of objects."Suppes (1960)

Accepting arguments (A) and (B) implies (C), thus explicating the often implicitly accepted position that set theory is the foundation of all mathematics. If correct, the argument implies that all mathematics is *reducible* to set theory and thus, in our terminology, is within a single conceptual framework, the CF of set theory. But as we will see in the next section, this argument is at least partially flawed, or at the very least misleading.

3 Why everything is not reducible to sets

3.1 Maddy's argument and the argument of non-identity

The conclusion (C) in the previous section combines, at least partially, what Maddy (2017) calls *Epistemic Source* and *Metaphysical Insight* in the discussion of set theory's foundational role in mathematics. According to *Epistemic Source*, set theory is used for "checking" whether some mathematical statement holds or not by checking whether it is provable in ZFC (cf. Section 6.1). *Metaphysical Insight* is the view that "the set-theoretic reduction of a given mathematical object to a given set actually reveals the true metaphysical identity that object enjoyed all along". Maddy (2017) argues that these views are flawed, because mathematical knowledge appears to be independent of set theoretical knowledge. Commenting on the claim "[T]he problem of mathematical knowledge reduces to the problem of knowing the set theoretic axioms", she writes:

"The trouble with this picture is that it's obviously false: our greatest mathematicians know (and knew!) many theorems without deriving them from the axioms. The observation that our knowledge of mathematics doesn't flow from the fundamental axioms to the theorems goes back at least to Russell – who emphasised that the logical order isn't the same as the epistemological order, that the axioms might gain support from the familiar theorems they generate, not vice versa..." Maddy (2017)

Maddy (2017) isolates also other ways in which ZFC, and set theoretic foundations in general, are perceived as a foundation for mathematics. These include: *Meta-mathematical Corral* which means that set theory provides the means to investigate provability and consistency claims made in mathematics; *Elucidation* which means that a set-theoretic replacement of an imprecise notion makes it precise; and *Shared Standard*, meaning that "if you want to know whether or not a so-and-so exists, see whether one can be found in [the universe of sets]; if you want to know whether or not such-and-such is provable, see whether it can be derived from the axioms of set theory." In the latter Maddy does not mean ontological, Platonic, existence, but rather a means of reaching practical agreement between mathematicians. She also proposes a new potential role for a foundational theory which, according to her, is not enjoyed by set theory (or category theory). She calls this *Essential Guidance*: "to reveal the fundamental features – the essence, in practice – of the mathematics being founded, without irrelevant distractions; ... to guide the progress of mathematics along the lines of those fundamental features and away from false alleyways."

If we look at these different roles for set theory as listed by Maddy, we notice that in most of them it is implicitly assumed that the set theoretic universe can be used to express mathematical notions other than sets.For example, in *Epistemic Source*, how can we prove anything about, say, knot theory in ZFC, if knots are not thought to be expressible in the language of set theory and their existence proved in the universe of sets? *Metaphysical Insight* explicitly states that not only is reduction to sets possible, but that it must, in fact, be done to know what mathematical objects are like. In *Shared Standard*, how can we "find" objects like continuous functions in a universe which contains only sets if there is no way to express them as sets? Or how can we prove something *about* Fourier transform in ZFC, if ZFC only contains axioms about sets and not about Fourier transform? Even in *Meta-mathematical Corral*, which seems to be neutral with respect to the identity of the objects, expressibility is the key requirement. In this view, constructing Dedekind cuts, for example, serves to verify that the existence of a complete ordered field is *consistent* with set theory.

However, it is important to note that being able to express a mathematical notion in terms of set theory is not necessarily enough for capturing, grasping and communicating that notion. In the case of Dedekind Cuts, for example, how do we know that the construction resulted in an ordered field? An ordered field is a model of the vocabulary $\{+, \cdot, <\}$, not of the set theoretical vocabulary $\{\in\}$. Thus one has to define sets that represent those symbols $\{+, \cdot, <\}$, sets that represent the entire language of ordered fields, and sets that represent semantic evaluation functions. Another option is to treat this part of the argument on a meta-mathematical level, but this does not change the fundamental difficulty. In both cases the meaning and conceptual information about orderings, fields and reals is *imported* from outside of the set theoretic framework.

3.2 Thought experiments: aliens and children

Consider the following thought experiment. Imagine that we had to communicate our mathematical ideas to an intelligent alien species. These aliens are so different from us that we share no sensory organs with them (they might only use gravity-wave echolocation, for example, and be able to detect changes in the intensity of γ-radiation). Furthermore, suppose that we have somehow managed to establish a common ground and share the basic mathematical notions of sets, first-order logic and the axioms of ZFC. What a "set theoretical reductionist" must accept now is that we are ready to communicate all about Hilbert spaces, finite automata, knot theory, Fourier analysis, variational calculus, Lie groups, integration on Riemannian manifolds and $\aleph_2$.

But this seems highly optimistic, if not downright impossible. To see why, imagine vice versa that the aliens start bombarding us with ever longer formulas in the language of set theory and we can check with our computers that the next is always indeed a consequence of the previous one. Even if the set theoretical derivations can be thus followed and checked, can we be sure that we understand what they are talking about? Let us keep in mind that their set theoretical constructions may involve equally sophisticated theories that we have, but on mathematical topics we are completely unaware of. A more likely scenario is that we would have no clue of what they were trying to communicate. The reason for this is simple: the conceptual framework we are using is that of set theory, while we do not know what the CF of the aliens is. We can communicate knot theory, for example, because we have the corresponding CF. But we cannot be sure that the aliens understand that CF. Even if they understood the set-theoretical construction, without any prior connection to knots, they might not be able to grasp what is being communicated.

Another example stems from a more down to earth scenario. Consider teaching natural number addition to children through set theory. It can be explained to a child what a set theoretic union is: given sets A and B, form the union $C = A \cup B$. If A and B are disjoint, say, A is a set of apples and B a set of bananas, then $|C| = |A| + |B|$. Thus one may jump to the conclusion that if a child understands the concept of set theoretic union, she then understands addition of natural numbers. This conclusion is natural if the concept of addition is thought to be reducible to that of disjoint union. This conclusion is nonetheless too fast, as was learned the hard way in the "new math" experiment of the 1960's and 1970's (Kline (1973)). The transfer from set theoretical language to that of arithmetic was not as smooth as the proponents of new math were assuming. In the end, new math was abandoned and arithmetical education was resumed from a non-set theoretical basis, i.e., in a different conceptual framework.

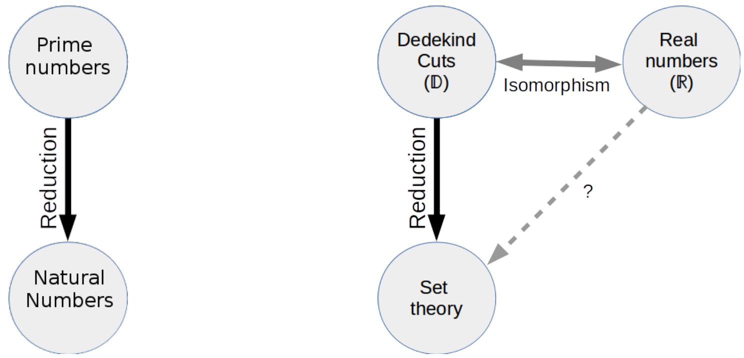


Figure 1: Left: Prime numbers are defined in terms of natural numbers and their operations. If the concept of $(\mathbb{N}, +, \cdot)$ is well understood, then the definition of the prime numbers is sufficient to capture the concept. **Right**: Concept of real numbers is not reducible to set theory. Not even via the isomorphism, because the isomorphism does not provide a reduction of the conceptual framework. It has to be created separately, see Section 6.2.

4 Some examples of reducible and irreducible mathematical content

Before we get into the details of reducibility and irreducibility of conceptual frameworks, let us first look at some cases in which it is easy to see that a new mathematical notion is reducible to earlier notions, as well as cases in which this is not the case. We will see that the key difference is in whether the conceptual framework that is used is the same or not.

4.1 Reducibility: prime numbers and the power set

The definition of a prime number in arithmetic is a good example of reducibility of a new notion to earlier notions. The definition is as follows: "a prime number is a natural number greater than 1 which is divisible only by 1 and by itself". This definition uses a variety of existing concepts and relations to define a new one. These ground notions are referred to by the terms "natural number", "greater than", "divisible" and "1". If all these prior concepts are understood, then they can be integrated in the described way and a new mathematical concept, prime number, naturally arises. In this way, the meaning and content of the expression "prime number" is reducible to a certain (simple) combination of these more basic concepts, see Figure 1(Left). Importantly, the conceptual framework (that of arithmetic) remains the same throughout the process.

Another example of reducibility comes from the concept of power set. Supposing that we understand the concepts of set and subset (and some predicate logic), we can put these concepts together as "the set of all subsets of a given set". Thereby the concept of power set is reduced to the concepts of "set" and "subset", employing the same conceptual framework of set theory.

4.2 Irreducibility: real numbers and Dedekind cuts

The easiest example of irreducibility is perhaps the fact that the basic notions, such as sets in set theory, cannot be reduced to anything more simple, because if they always could be, we would face infinite regress.

But there are also more interesting cases of irreducibility in mathematics, starting from the familiar construction of real numbers in set theory. Can the idea of real numbers be reduced to the mere idea of a set? The standard attempts (Dedekind cuts, Cauchy sequences etc.) give rise to isomorphic structures, which may seem to suggest that real numbers are reducible to set theory. However, it is important to note that a construction like Dedekind cuts is done in a different conceptual framework. Dedekind cuts are invented to satisfy an intuition about real numbers, not vice versa. The properties that we attribute to real numbers, such as continuity and being an ordered field, can be seen as properties of Dedekind cuts only in an *emergent* way. This is in fact analogous to the way "freezing" can be said to be a property of water, but not of water molecules let alone quarks, as we will argue in detail in Sections 5 and 6. In Section 6.2 we will also argue why the existence of an isomorphism does not constitute a reduction, Figure 1(RIGHT).

Here is what Maddy has to say about Dedekind cuts:

> "...a Dedekind cut is a set of rationals, which are equivalence classes of sets of pairs of natural numbers, which are ordinals, and so on, but none of this detail has any direct connection to their intended behavior as surrogates for the reals, as the availability of alternatives like Cauchy sequences serves to demonstrate." Maddy (2017)

She mentions "intended behavior" as something fundamental to real numbers. However, this "intended behavior" is justified *outside* of set theory, in a different conceptual framework. Nevertheless, the intended behavior is integral to what we refer to by "meaning" and "conceptual content". Thus, although the set theoretic reduction of real numbers can be done in purely formal terms, it does not follow that the conceptual content of real numbers can be reduced to the conceptual content of sets. The difference in the respective conceptual frameworks makes this problematic. A little different type of an example is given by the Bolzano theorem, see Section 5.[1]

[1] Yet another interesting example of such phenomenon is the construction Lebesgue measure. One starts by defining the length of open intervals proceeding inductively to Borel sets through countable limits in order to arrive to something that is originally motivated through a collection of seemingly unrelated intuitions about mass density and probability distributions as well as mathematical principles such as σ-additivity.

5 Epistemological reducibility and emergence

When it comes to the meaning of mathematical notions, what is the opposite of reducibility? If the meaning of a concept is not reducible to earlier concepts, it has to come from something that is not present in the conceptual framework of the earlier concepts. Although real numbers can be constructed as Dedekind cuts, their intended meaning can not be derived from this construction. It has to come from somewhere else. But what can this mean? After all, the common view in mathematics is that meaning *always* comes from definitions, such as real numbers defined as Dedekind cuts.

To make sense of the emergence of meaning of mathematical concepts, let us make an analogy by referring to the discussion of reducibility and emergence in philosophy. There are a great deal of different views on what is meant by emergence and reducibility. The usual starting point is that emergent properties arise from lower-level properties, but are not reducible to them. In this way, reducibility can be seen as contradicting emergentism. In some cases, the distinction between reducibility and emergence is indeed quite clear. Under a very strict notion of reducibility, a concept or a phenomenon A is *reducible* to a set of other concepts or phenomena $F_1, F_2, \dots$, if A is *identical* to $F_1, F_2, \dots$ taken separately. For example, the concept of "two pens" is reducible to "one pen" and "one pen". It is clear that "two pens" is not emergent in any interesting sense, but there are cases in which the question of reducibility and emergence is less clear.

Perhaps the best candidate for "the standard view" on *ontological* emergence is the so-called "supervenient emergentism". Under this understanding of emergence, if higher-level emergent properties are altered, then by (nomological) necessity so are some of the lower-level properties (Van Cleve (1990); McLaughlin (1997)). Thus supervenient emergentism is committed to downward causation: higher-level causes have lower-level effects. O'Connor (1994) defines *strong emergent properties* accordingly: they are properties of a complex object that supervene on the parts of the object, are not shared by any of those parts, are not structural properties of the complex, and have downward causation to the parts of the object. There is a great deal of debate concerning this kind of emergence among philosophers philosophy, see e.g. Kim (1999); Kallestrup (2006); O'Connor (1994). We cannot go into the details of that debate here, but the controversial character of this kind of strong emergence has led many philosophers to look for less problematic views on emergence.

Perhaps the best-known of these has been presented by Bedau (1997) and Clark (1998). According to them, to contrast the strong supervenient emergentism, we should also look for weaker versions of emergence. They point out that the interactions in the lower-level states can be so complicated that reducing the higher-level phenomena to them may be in practice impossible, thus making an exhaustive explanation unfeasible. Well-known examples of this are chaotic systems, whose higher-level dynamics are very sensitive to minuscule lower-level details. But to contrast this kind of weak emergentism with the strong ontological emergentism, in Bedau's definition of weak emergence it is *in principle* possible to derive the higher-level dynamics by simulating the lower-level phenomena. Thus weak emergence is not necessarily in contradiction with reducibility. The higher-level dynamics of the system can in principle be reduced to the lower-level phenomena so they are not emergent in the strong sense of supervenient emergentism.

To give an example, as of writing this, the "Mpemba effect" in which warm water can freeze faster than cold water is still in need of an explanation. Is freezing an emergent property of water? In the strong sense, this seems implausible, since no downward causation is likely to be involved: according to the standard story in chemistry, the causal activity takes place only on the level of molecules (and ultimately the quantum-physical properties of the molecules). In the weak sense, however, emergence is in this case a much more appealing position. At least currently, it is not known how to reduce freezing to known (quantum-) physical properties of water molecules. Thus one can coherently believe both in reduction and emergence of the same phenomenon at the same time: we may believe that the freezing of water as a macro-level phenomenon can ultimately be reduced to its micro-level parts, but we can also believe that our current limits in understanding the phenomenon make freezing an emergent property. This is possible because the relevant notion of emergence is a weak one.

Another way to understand the difference between strong and weak emergence is to see them, respectively, as ontic and epistemic notions see, e.g. Povich and Craver (2018). On an ontological level, the macro-level properties of water are thought to be determined entirely by the micro-level properties of water molecules. On an epistemological level, however, we may see some macro-level properties as emergent since they demand modes of explanation that are not applicable to the micro-level properties.

Sometimes this kind of epistemological emergence may occur due to computational complexity. To see this, let us consider the "two-body" (2BP) and the "three-body" (3BP) problems in physics. If there is just one body, its dynamics are a straight-forward consequence of Newton's first law of motion, as the body progresses linearly in time. If we do not assume any gravitational interaction between the bodies, the same will be the case with two or three objects. As soon as gravity is included, however, the problem becomes quite different. Whereas the two-body problem can be solved and we can determine the motion of the bodies given initial conditions, with three bodies the problem of predicting their behaviour becomes intractable for most initial conditions (Herman (1998)). This behaviour, however, is still ontologically reducible to the properties of the bodies and their gravitational interactions. So in the ontological sense, there can be no difference between the 2BP and the 3BP. But epistemologically there is an important distinction.

Example	Epistemologically emergent	Ontologically emergent
Two pens	No	No
2BP	No	No
3BP	Yes	No
Mpemba	Yes	No
Prime number	No	No
$\mathbb{R}$	Yes	No

Table 1: Examples of non-mathematical phenomena (above) and and mathematical ones (below) and their status in the epistemological and ontological emergence frameworks. Although it may seem that mathematical definitions are more analogous to "two pens" than to "3BP", we contend that the cases of prime numbers and the real number differ in terms of epistemological emergence (and thus reducibility) too.

Table 1 shows an overview of epistemological emergence and reducibility through the above examples. It may seem strange that the case of two pens is similar to the two-body problem. Of course this is not to suggest that the problems are equally difficult. Determining whether there are two pens in a collection is a trivial problem whereas 2BP is not, but for our approach in this paper, the two problems are essentially the same. The important aspect here is that solving a case of 2BP does not require bringing in higher-order methods: it can be solved by the same principles we use to explain gravity and the movement of objects. In this way, while 2BP is in practice more difficult than the one-body problem, it is not essentially different from it. With the 3BP, it is not only the practical difficulty of the problem that changes. Rather, we need a change in the whole mode of explanation (bring in the language of approximations, chaos theory, topology and so on).

6 Reducibility and emergence of mathematical concepts

Now that we have established the context of reducibility and emergence, we would like to apply it to mathematics, namely to answer Question 3 of the introduction: *What exactly are those other ways of creating mathematical content, different from formal definitions?* In this section we argue that the context of reducibility and emergence can be fruitful to the discussion about mathematical concepts.

As mentioned in Section 3 "imported" properties of mathematical objects such as Dedekind cuts tend to appear when their definitions are motivated or otherwise guided by principles and intuitions "from outside". By "imported',' we mean properties that are not part of the conceptual framework in which the concepts are defined. Dedekind cuts are defined in set theory, but we attribute to them notions like are "continuity", "being a field", "points on a line" etc. This is analogous to the way in which emergent properties such as "freezing" are attributed to water while we maintain that (ontologically) water is composed of water molecules. The property of freezing does not apply to the water molecules, which can be neither frozen nor melted, but water nevertheless consists of the molecules. Similarly, real numbers can be expressed in the set theoretical universe with the construction of Dedekind Cuts. But in the conceptual framework of set theory, we do not discuss notions such as "continuity". Instead of the CF of set theory, they are imported from another CF.

This is of course not always the case when introducing new concepts in mathematics. Concepts like prime numbers are conceptually completely exhausted by their definitions once the CF of arithmetic is assumed to be understood. This is analogous to the 2BP in which the orbits can be epistemologically and operationally reduced to the Newtonian equations. No part of another CF needs to be imported, and as such the new notion is reducible to the earlier concepts of the CF.

This matter is not limited to introducing individual new concepts. More generally, entire CF's can emerge from complex interactions between other CF's. Alternatively, new CF's can come about in some other ways. Our account does not claim to include all the possible ways in which this could happen, but we propose some ideas in Section 7.

6.1 Descriptions and constructions

Philosophically, we can make a distinction between two types of mathematical definitions which we call *descriptions* and *constructions*.[2] Whether a mathematical definition is one or the other is predictive of whether the defined object will enjoy emergent properties. In the philosophy of mathematics, many approaches (e.g. Nagel (1935); Shapiro (1997)) take constructions to be one method of introducing new entities to mathematics (alongside stipulations and implicit definitions). Constructions typically introduce entities which have *intended* properties that are not inherent to the CF in which they are constructed. Thus constructions require importing content from another CF. Descriptions, on the other hand, are characterised by a reduction of the CF.

To give a simple example, the definition of π in the form "the ratio of the circumference of a circle to its diameter" is a description while the (mathematically equivalent) definition

$$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \tag{2}$$

would be a construction.

Referring back to Section 3, why can't we expect the aliens to understand our set theoretical constructions as the rich mathematical concepts they intend to be? Now we can state

[2]We do not intend to refer to the constructivist approaches in the philosophy of mathematics where the only valid constructions are those that do not rely on the law of excluded middle or the axiom of choice (Troelstra and van Dalen (1988)).

the reason clearly. If we only have shared the notion of set and membership with them, then all we can reliably communicate to them is in the CF of set theory. If we show them the Dedekind construction, they will see infinite sets of ordered pairs, not real numbers. Reals are in a different CF and thus epistemologically *emergent* in set theory. In the same way, as if we only knew Newtonian mechanics, we would be able to solve the precise orbits in the 2BP, but not even begin to understand the case of the 3BP.

Why was it naïve to assume in the New Math development that if a child understands disjoint union, she will readily understand addition of natural numbers? Under our account, it becomes clear that the notions of "addition" and disjoint "union" reside in mutually irreducible CF's. To establish a translation between them one needs to have both of these CF's in the first place.

This distinction between different conceptual frameworks is crucial for explaining how mathematical concepts are captured, grasped and communicated. The problem in reducing mathematics to set theory is that this difference is not recognized. Under a purely formal approach, the construction of Dedekind Cuts can be equated with real numbers. This may lead to the mistaken impression that real numbers are reducible to set theoretical concepts. But if we had the CF of set theory only, we would probably never get the idea of carrying out this construction. This is a crucial point: that a mathematical concept C is *expressible* in some formal theory F does not imply that the concept is epistemologically *reducible* to the (intended) conceptual framework of F.

This can be illuminated by a simple example from calculus. Bolzano's theorem states that every continuous function from reals to reals which attains values a and b, attains also all the values between a and b. This is not a statement which is classically thought of as independent of ZFC (such as the Continuum Hypothesis). In fact, if real numbers are Dedekind cuts and if continuous functions are sets of pairs or real numbers satisfying certain set theoretic formulae, then Bolzano's theorem can indeed be proved in set theory. But how does this square with our claim that there is more to real numbers and continuous functions than just sets and that set theory can only talk about sets, but not about real numbers?

Technically, this seems to imply that Bolzano's theorem, taken in a wider context than formal mathematical theories, is not provable in ZFC. In fact, ZFC cannot even talk about it, because it is in a different CF. We can *interpret* Bolzano's theorem as a provable statement in set theory. This fact is the celebrated feature of set theory, namely providing a unified formalisation for all mathematics. But strictly speaking, saying that Bolzano's theorem is provable in ZFC is like saying that the pipe in a painting of a pipe is a pipe.[3]

[3] We are referring to the famous 1929 painting by René Magritte *The Treachery of Images* which depicts a smoking pipe with the accompanying text *"Ceci n'est pas une pipe."*, French for *"This is not a pipe"*.

6.2 Why is isomorphism not a reduction?

Perhaps the biggest counter argument to the existence of mutually irreducible conceptual frameworks in mathematics is the argument that since real number line is *isomorphic* to the system of Dedekind cuts, it is reducible to it.' In this view, Dedekind cuts can be identified with the real numbers because there is provably an isomorphism between any two complete non-trivial ordered fields. Under this view, once we present the system of Dedekind construction to the aliens (Section 3.2), they will recognise it as isomorphic to something they know which in turn would be very close to what we mean.

However, there is no reason to regard an isomorphism as carrying a reduction of the CF from real numbers to set theory, see Figure 1(Right). The isomorphism is an extra step which cannot be read out from the set theoretic concepts and their interactions.

Here it is important to emphasise that we use *epistemological* emergence as an analogy, not *ontological*. Had we taken the ontological approach, it would be feasible that the real number line is reducible to sets. To be more precise, this would be the case if we accepted an ontological version of *structuralism*. According to structuralism, mathematical ontology does not concern mathematical objects (such as numbers or sets) but rather mathematical structures such as Peano Arithmetic or ZFC set theory (Shapiro (1997)).[4] Under the structuralist interpretation, if A is isomorphic to some system C that is emergent from B, then A and C are the *same* system, and thus also A would have to be emergent from B. With an ontological understanding of structuralism, there would be no difference between the systems A and B. But with the epistemological approach, this changes. B alone does not provide the conceptual framework for A, and as such A is not reducible.

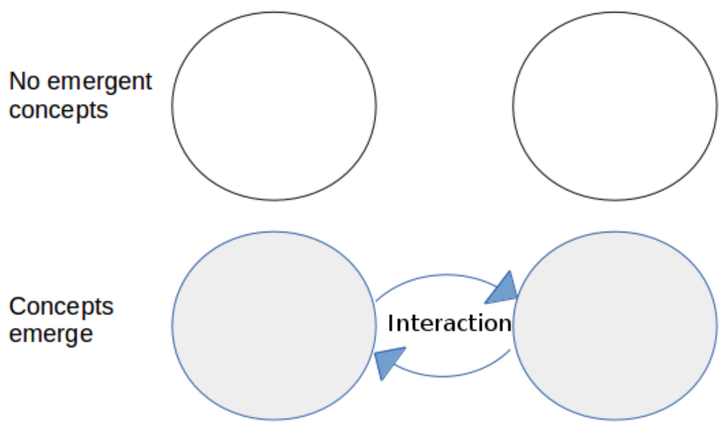
It is this epistemological approach to mathematical concepts and their meanings that we are interested in. Thus we see the existing foundational theories in mathematics lacking in an important way. They are only concerned with the ontological level, i.e., whether a mathematical concepts can be formulated in terms of other mathematical concepts. But a satisfactory foundational theory of mathematics ought to explain more than that. It would not only explain how mathematical concepts can be constructed, but also give an account on how the emergence of new concepts and new conceptual frameworks takes place. Equally importantly, it would also explain what the mechanisms are by which the meaning, content and understanding of these CF's come to being.

[4]We do not refer to the model theoretic notion of a structure, but the philosophical notion put forward by Shapiro (1997): "A *structure* is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system". Emphasis in the original.

7 How do new conceptual frameworks emerge?

Since our contention is that mathematical concepts are not always epistemologically or content-wise reducible to lower-level concepts, this prompts the question what the emergence of mathematical concepts come down to. If new conceptual frameworks can be created without the philosophically questionable postulation of “unexplained” basic principles, how exactly does it happen? As a special case, how is our intuition of set theory itself grounded and where does the meaning and content of the set theoretic expressions such as power set and union come from?

Our proposal is that when the interaction of concepts from many different CF’s is sufficiently complex, new ways of thinking are necessary to comprehend the emergent conceptual system. This can include postulating new concepts whose content and semantics is grounded in the *interactions* of previously defined concepts, as well as their CF’s (cf. Kulikov (2015)). The real number line, for example, is a concept which is supervenient on the CF’s of geometric intuition, arithmetic, set theory and mathematical analysis. Each of those separately can only provide an incomplete picture of the abstract concept.



7.1 Analogy to Hesperus and Phosphorus

Frege (1948) famously made a distinction between the sense and the reference of a concept. In his example, it was noted that the concept “Morning star” (Phosphorus) has the same reference (the planet Venus) as the concept “Evening star” (Hesperus), but they have different senses Frege (1948). Before it was known that the two concepts refer to the same object, they had developed different meanings. The two concepts were connected with the discovery of the shared extension, but they still retained the different intensions. In terms of conceptual frameworks, this development could be described as follows:

1. First there was the CF of constellations of light-dots (stars) and their relative locations

in the evening, morning and the night sky. Some stars would appear in less coherent locations relative to the other ones, some in the morning, some in the evening, some throughout the night. Two of those, one appearing in the morning, one in the evening were given names Phosphorus and Hesperus, respectively.

2. Along with the development of science and perhaps new tools, new CF developed such as one in which there are actual objects far away in the cosmos.

3. Once these two frameworks are connected by realizing that the far away objects are the ones we see as light-dots in the sky, it was possible to conclude that Hesperus and Phosphorus are in fact the same heavenly body.

Thus within the original CF of dots in the sky and their relative positions, it is impossible to know that concepts Hesperus and Phosphorus have a shared reference. It is possible to speculate on some relation between the two because of their similarities (e.g. direction of movement), but to grasp that the shared reference one must shift attention to another CF in which there are far away objects that have orbits in the cosmos. Moreover, it is necessary to bridge these two CF's to one another by some method.

This is analogous to how the meaning of mathematical concepts is determined. The idea that the number line and the collection of Dedekind cuts are both representations of the same thing requires bridging together the corresponding CF's. Only through this kind of process can we create a new CF in which it is possible to determine that they have the same reference, even though their senses are different due to being in different conceptual frameworks.

7.2 Analogy with time

Another illustrative example is that of time. According to the International System of Units SI, one second is a period of time that is equivalent to $9,192,631,770$ cycles of a Caesium atomic clock.[5] Most people who use seconds on a daily basis, however, probably do not know that. In some sense, Caesium atomic clock to seconds is like set theory to real numbers. Another, more familiar definition of a second, would be that it is $1/(24 \times 60 \times 60) = 1/86,400$ of a day. Let us consider a simple thought experiment. Imagine that tomorrow we will wake up and notice that the length of the day and the length of the cycle of a Caesium atomic clock do not match the way they did before. Namely, if measured by the Caesium atomic clock, the day is no longer $86,400$, but $86,412$ seconds. Which of the two conclusions will we jump to: (1) the day became longer for whatever reason, or (2) the cycle of the atomic clock became shorter for whatever reason? One way to solve this question is to compare both of them to some third-party definition of a second. For example, if the cycle of Caesium atomic

[5]https://www.bipm.org/en/publications/si-brochure/second.html

clock was known to be exactly twice as long as the cycle of XYZ atomic clock, we could compare Caesium to XYZ. If Caesium atomic clock cycle is *still* twice as long as the one of XYZ clock, then it seems that the day got longer. But if the ratio of Caesium cycle to XYZ cycle is now shorter, then we know that the day remained as long as before, but something happened to the Caesium atoms.

Imagine, however, that all known ways of measuring time have gone out of synchronicity. The ratio of XYZ cycle to Caesium cycle has changed, the ratio of Caesium to day-length has changed and moreover, the ratio of XYZ to day-length has also changed. How can we now know the answer to the question whether the day got longer or the Caesium clock sped up, or perhaps both happened? What is particularly interesting is how we should now redefine "one second" so that the duration of the new second is exactly as long as that of the old second? It seems that it is in fact impossible, and moreover an ill-defined question. It is ill-defined, because the notion of the old-second didn't *really* depend on Caesium atomic clock, but rather on the ratios of all the different frequencies we are surrounded by and crucially on the robustness of these ratios (robustness in the sense that *most* of the ratios wouldn't change at the same time). This thought experiment reveals that even though the official definition of a second uses a particular way of measuring time, it is not the *true* notion of a second. The true notion, rather, is grounded in a multitude of ways to measure time.

This is analogous to Dedekind cuts not being the *true* definition of real numbers even if for mathematicians it may serve as the prototype of them. In our terminology, the CF in which the concept "one second" resides is an emergent CF. It emerges out of the interaction of many other CF's each of which provides us only with incomplete attempts to understand the abstract concept of "one second". A similar thing happens in the emergence of new CF's in mathematics.

7.3 Parallels in cognitive science

In this section we present three *cognitive* theories from the literature which have a lot in common with our approach.

7.3.1 Metaphor blending

Expanding on the origins of number concepts, Lakoff and Núñez (2000) have proposed that the understanding of mathematical concepts is grounded in embodied representations, and the conceptual frameworks are transferred via metaphor maps. In terms of conceptual metaphor theory of Lakoff and Johnson (1980), these maps transfer concepts and structure from a source domain (that we are familiar with) to a target domain (that we are trying to explain). In this way, for example, the experience of comparing lengths or distances (source domain) is metaphorically mapped onto the real line (target domain), which enables us to understand the relations "<" and ">" in mathematics. An interesting concept in this theory is the concept of a metaphor blend. This is where two "source domains" are blended to yield a new, possibly more abstract, target domain. An example is what Lakoff and Núñez call the *basic metaphor of infinity* (BMI, see also Núñez (2005)). The idea in BMI is that the notions of perfective and imperfective aspects get metaphorically mapped to the same, new, domain yielding the idea of actual infinity. For instance the verb *to jump* includes the idea of an ending as in "I jumped", but *to run* does not imply an ending as in "I ran". Metaphorically blending these two notions leads to a new conceptual framework where an activity is ongoing, but has a final state nonetheless. This theory provides a cognitive framework for how new conceptual frameworks can be created and how understanding those new frameworks can be facilitated.

The idea of a "domain" in the theory of metaphor blending is akin to our "conceptual framework". To translate to the language of metaphors, a concept is reducible in a CF if no new metaphors are needed to fully comprehend it based on its definition. If new metaphors are needed, then a new CF is created. The idea of grasping new concepts through metaphor *blending*, on the other hand, corresponds to our idea of emergence of concepts from interaction.

7.3.2 Processes

Pantsar (2015) has presented an alternative metaphorical approach to mathematical concepts, with the focus on *processes*. In this account, mathematical concepts (or objects) are seen as metaphorical counterparts of mathematical processes, which in turn have their basis in processes carried out in our physical environment. Natural numbers, for example, enter mathematical discourse as objects because they are the target domain for the metaphorical discourse of processes as objects, based on the source domain of the process of counting. Counting, in turn, is a process tightly connected in its origin to physical processes, such as, extending fingers or tallying.

Concerning real numbers and getting back to the Dedekind construction, one could argue that the concepts of the real number and the Dedekind Cut are cognitively different due to being the metaphorical counterparts of different processes. We use different metaphors and different processes in understanding the two, thus having different conceptual frameworks. The realisation of the isomorphism between them is a new cognitive step which ties them together, but this isomorphism is part of a new CF.

7.3.3 Learning to count

When we discuss the emergence of conceptual frameworks, one interesting case is the way in which children acquire natural number concepts. According to mainstream view, initially, children only learn to recite the numeral sequence, without being able to match it to quantities. At that stage, a child can be familiar with the verbal recitation “one-two-three-four-five”, yet she cannot use numerals to refer to quantities (Fuson (1988); Wynn (1990); Davidson et al. (2012)). In this way, children can acquire a syntactic placeholder structure without positing any semantic content to its members. This can be thought of as one CF and it consist only of a string of sounds, comparable to ”eenie-meenie-miney-mo”. It is only later that they gradually start grasping that members of the numeral sequence can be used to refer to discrete quantities. A popular explanation of how this happens is the process of “bootstrapping” (Sarnecka and Carey (2008); Carey (2009); Spelke (2011)). Taking learning in a holistic Quinean 1960 context, the bootstrapping theory proposes that from the existing web of representations that children have from processing quantitative information which can include for instance several proto-arithmetic abilities common to most mammals Feigenson et al. (2004) (i.e., other CF’s), semantic content emerges that enables saturating the placeholder structure with meaning. More specifically, starting from familiarity with a small group of numerals, children “bootstrap” an early form of the successor function, thus making them grasp the progression of natural numbers. An important stage in this is acquiring what Sarnecka and Carey (2008) call the “cardinality principle”, i.e., the understanding that every numeral on the list refers to a unique discrete quantity. It is natural to think of this process as an emergence of new abstract CF from a complex interaction of previous ones.

8 Discussion and conclusion

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